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International Centre for Mechanical Sciences

New Trends in Thin Structures: Formulation, Optimization and Coupled Problems

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 519

NEW TRENDS IN THIN STRUCTURES: FORMULATION, OPTIMIZATION AND COUPLED PROBLEMS

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PREFACE

The application area of thin structures covers a wide range in all parts of engineering. Hence considerable effort has been devoted over the last years to enhance mathematical models, methods and algorithms for nonlinear structural behaviour. The main focus is to convey modern techniques applied within the range of computational mechanics of beams, plates and shells. This CISM course brought together leading experts in the field of shell analysis, multifield methods and interface treatement to highlight the state of the art on mathematical, mechanical and engineering aspects of current finite element technology.

The topics of interest are wide ranging, and include the followng computational aspects of

1. *Nonlinear theory of shells and beams including dynamics*
2. *Advanced discretization methods for thin shells and membranes*
3. *Shear-deformable Shell Finite Elements for SMA composite devices*
4. *Optimization and design of shells and membranes*
5. *Fluid-Structure Interaction with thin-walled structures*
6. *Contact mechanics with application to thin structures and*
7. *Edge effects in laminated shells.*

The contributions cover the background of the fully nonlinear formulation of beams and shells and the associated finite element discretization of beams, thin shell and membrane structures undergoing large deformations ncluding new robust discretization techniques for shear-rigid shell models. In particular, the use of smooth shape functions for discretizing the curvature dependent thin-shell energy functional are discussed.

Another topic covers the design of membranes including follower loads as well as ideas regarding manufacturing of membranes including pre-cutting and blanking. Formoptimization of thin shells regarding maximum stiffness and optimization of creases are provided. In this line of topics also numerical methods are presented including stabilization methods for inverse problems with modifaction of geometry. Furhermore efficient and fast algorithms for sensitivity analysis are discussed.

Shape memory alloys are nowadays well known and widely adopted as active components for innovative actuators. Related to these developments possible modelling of thin structures made of SMA or SMA-based smart hybrid composites are presented. This includes the general discussion of laminated structures and three-dimensional analysis of edge effects.

Contact takes often place when shell or beam structures interact with each other. This can occur on structural level, like in car-crash analysis but also within the investigation of composite materials where fibers interact in laminates with the matrix material or woven fabrics are in contact state. For these problems new contact formulations are developed and formulated up to the level of discretization techniques and algorithms using finite element techniques.

In many advanced application scenarios structures interact with the surrounding or enclosed fluid (liquid or gas). Especially for thin-walled structures such interaction effects are often essential for the overall structural behavior and have to be taken into account. On the other hand thin-walled structures pose a number of challenges to coupled modeling and simulation approaches. These challenges are discussed along with appropriate solution approaches. In details methods for fluid-structure interaction are presented like interface tracking and interface capturing schemes. This includes important issues which relate to geometric conservation laws, computational mesh dynamics and distorting elements.

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Peter Wriggers

CONTENTS

A plate theory as a mean to compute precise 3D solutions including edge effects and related issues

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Abstract The question of the estimation of the quality of a plate or a shell theory with respect to the 3D reference solution has been the subject of intense research from the end of the seventies to he beginning of the eighties. This subject has been seen then more or less as an old fashioned one and the knowledge acquired in those years has been largely forgotten in favor of Finite Element approaches. It is however our opinion that a clear insight of the intimate relation between the exact 3D solution and the plate or shell one is still of interest. In fact, at least in the case of elasticity, a plate or a shell theory, completed by edge effects analysis, is an extremely powerful tool to compute a precise approximation everywhere.

1 Introduction

Much research has been devoted, especially in elasticity, to the definition and validation of the classical theories of plates and shells on the basis of the 3D reference model. An initial and very important body of work focused on the development of a posteriori error estimators which do not introduce so-called edge effects: [11], [6], [27], [12]. Error estimators which include edge effect corrections were given in [13]. Thus, by the end of the 1980's, the classical theories of plates and shells, i.e. the theories of Kirchhoff-Love and Reissner-Mindlin, were relatively well-understood in the framework of linear elasticity. Another body of work is related to the technique of asymptotic developments, with the thickness as the small parameter: [9], [10], [5], [8], [7], [3], [4]. More recent work was aimed at improving the classical theories; it was demonstrated that, with few modifications, the Reissner-Mindlin theory turns out to be a 2nd-order theory [13], [16], [14]. Still other work concentrated on the development of refined theories, particularly for composites [23], [28], [22], [25]. All the work mentioned above, without exception, was based on the premise that a theory of plates or shells remains an approximation which can be satisfactory only if the thickness is relatively small. Another view developed in particular by Ladevèze is that in the calculation of beams, plates and shells, which has been so far developed for plate in the case of isotropic elasticity in, is that the solution can be partitioned into two parts:

- one, called Saint-Venant's solution, in which the variation length is large, and
- one which contains the phenomena with a small variation length, which are, therefore, localized.

The concept of Saint-Venant's principle is the basis of this approach. It could, as in the case of beams, suggest a theorem which specifies the conditions on the data which would ensure that the solution is localized in the vicinity of the edge or edges. This unitary vision of Saint-Venant's principle was presented in [15], where it was demonstrated that this localization occurs if, and only if, the solution associated with the particular data is orthogonal to a distinctive family of solutions which, in the case of beams, is Saint-Venant's classical family of solutions and, in the general case, is the family of solutions which describes the phenomena with a large variation length. Both can be calculated from a plate theory which is, formally, quite close to the Kirchhoff-Love theory. Consequently, this plate theory is exact, since the generalized quantities it generates are exact that is coincide with the one extract from the exact 3D solution [18]. In the case of beam this approach has given rise to the development of precise and efficient tools for the calculation of the 3D solution [20], [21], [2].

This approach will not be discussed furthermore in this paper. We will follow the line of [17], but up to the first order only. We will concentrate mainly on the reconstruction of the 3D approximation from the one obtained by a plate theory. For the inner part of the solution this approximation, can be interprted as an approximation of the St-Venants's or long wave length solution. In order to be valid everywhere this approximation should be completed in order to incorporate possible approximation of the edge effects, if they are important. A key issue then is that the problem deduced from the residual of plate theory on the edges should lead to a localized problem. This means that its computation can be reduced to a computation of the edge region only. If not, this computation would require a 3D problem on the whole body, which would lead to a modification of the interior approximation of the 3D solution of the same order that the approximation deduced from the plate theory itself. This in turns would mean that the plate solution itself is meaningless, at least for the order of magnitude which is considered. This last point should be considered for any plate or shell theory, otherwise the improvement that is expected from improved shell theory is, at least, debatable. We will concentrate on the exploitation of the classical plate theory, i.e. the Kirchhoff-Love one, in the case of composite laminated plates constituted of a stacking of orthotropic layers. The main aspect of the theory will be first presented, then reconstruction of the approximation of the 3D interior solution will be described. For sake of simplicity we will go further only in the case of isotropic plate for which we will developed the reconstruction up to an approximation which verify all the equation but the edge conditions, which will be satisfied in a mean sense only. The construction of the approximation of the edge effects will be developed before the conclusion.

2 The classical theory of Laminated Plates or CLT

2.1 Description of the plate and of the constitutive relation

The geometry of the plate is described through its mid surface Σ_m it thickness, $2h$, which will be assumed to be constant. A point M of the plate, has an orthogonal projection m on the mid surface and its coordinate along the normal $\underline{z}$ to the mid surface is denoted by z:

$$M = m + z\underline{z} \tag{1}$$

Let us also note Σ^+, Σ^- and Σ_l the upper, lower and lateral surfaces of the plate (see figure 1).

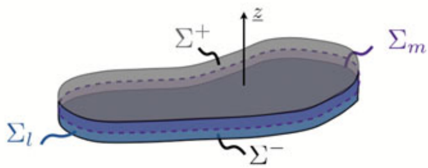

Figure 1. Plate geometry

In the case of the strain and the stress tensors, we will make use of the following notations (described below in the case of stresses):

$$\underline{\underline{\widetilde{\sigma}}} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \tag{2}$$

$\underline{\underline{\widetilde{\sigma}}}$ is the in plane part of the stress tensor. The antiplane part of those tensors is denoted by:

$$\underline{\underline{\sigma}} \cdot \underline{z} = \left\{ \begin{array}{c} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{array} \right\} = \left\{ \begin{array}{c} \underline{\sigma}_c \\ \sigma_{zz} \end{array} \right\} \tag{3}$$

The orthotropic constitutive relation of each ply of the plate can be written as:

$$\left\{ \begin{array}{l} \underline{\underline{\widetilde{\sigma}}} = \widetilde{\mathbb{K}}\, \underline{\underline{\widetilde{\varepsilon}}} + \mathbb{A}\, \varepsilon_{zz} \\ \underline{\sigma}_c = \mathbb{B}\, \underline{\varepsilon}_c \\ \sigma_{zz} = C\varepsilon_{zz} + \mathrm{Tr}\left(\mathbb{A}\, \underline{\underline{\widetilde{\varepsilon}}}\right) \end{array} \right. \tag{4}$$

where the linear operators $\widetilde{\mathbb{K}}$, $\mathbb{B}$, $\mathbb{A}$ and the scalar C vary through the thickness in the case of laminates. In the orthotropic basis of each ply, they take the following form:

$$\widetilde{\mathbb{K}} = \left[\begin{array}{ccc} K_{11} & K_{12} & 0 \\ K_{12} & K_{22} & 0 \\ 0 & 0 & 2G_{12} \end{array} \right] \tag{5}$$

$$\mathbb{B} = \left[\begin{array}{cc} 2G_{13} & 0 \\ 0 & 2G_{23} \end{array} \right] \tag{6}$$

$$\mathbb{A} = \left[\begin{array}{cc} A_1 & 0 \\ 0 & A_2 \end{array} \right] \tag{7}$$

Introducing the Hooke tensor under plane stress condition $\mathbb{K}_{cp}$, one obtains:

$$\underline{\underline{\widetilde{\sigma}}} = \mathbb{K}_{cp}\, \underline{\underline{\widetilde{\varepsilon}}} = \widetilde{\mathbb{K}}\, \underline{\underline{\widetilde{\varepsilon}}} - \frac{\mathbb{A}}{C}\, \mathrm{Tr}\left(\mathbb{A}\, \underline{\underline{\widetilde{\varepsilon}}}\right) \tag{8}$$

Taking this relation into accounts, the inverse of the constitutive relation of the ply can be written as:

$$\left\{ \begin{array}{l} \underline{\underline{\widetilde{\varepsilon}}} = \mathbb{K}_{cp}^{-1}\, \underline{\underline{\widetilde{\sigma}}} + \mathbb{A}'\, \sigma_{zz} \\ \underline{\varepsilon}_c = \mathbb{B}^{-1}\, \underline{\underline{\sigma}}_c \\ \varepsilon_{zz} = C'\, \sigma_{zz} + \mathrm{Tr}\left(\mathbb{A}'\, \underline{\underline{\widetilde{\sigma}}}\right) \end{array} \right. \tag{9}$$

with:

$$\left\{ \begin{array}{l} \mathbb{A}' = -\dfrac{1}{C}\, \mathbb{K}_{cp}^{-1} \mathbb{A} \\ C' = \dfrac{1}{C}\left(1 - \mathrm{Tr}\left(\mathbb{A}\mathbb{K}_{cp}^{-1}\mathbb{A}\right)\right) \end{array} \right. \tag{10}$$

2.2 Basic assumptions of the CLT

The in plane displacement (parallel to Σ_m) are denoted by $\underline{V}$ and the normal part of the displacement (parallel to $\underline{z}$) by w. The approximation of the displacement in the Kirchhoff-Love theory is:

$$\underline{U}(m,z) = \underline{V}(m) + w(m)\underline{z} - z\,\underline{\text{grad}}_m\,(w(m)) \tag{11}$$

This approximation states that a segment normal to the mid plate has a rigid body displacement and stays normal to the mid plane in the deformed state. This approximation should not be considered to be valid for the derivation of its gradient with respect to z. In fact the thickness being small, a small error in the displacement, when derived with respect to z, leads to a large error in the gradient, this explains what is often call the plate paradox. On contrary one makes use of a complementary static hypothesis (which will be justified later on), which is that the normal or peeling σ_{zz} is negligible with respect to the other components of the stress tensor. In order to make use of those mixed hypothesis, it is convenient to make use of the Hellinger-Reissner mixed formulation which states that the solution $(\underline{U}, \underline{\underline{\sigma}})$ of an elastic problem under the small displacements assumption, is the stationary point of the HR$(\underline{U}, \underline{\underline{\sigma}})$ functional , with:

$$HR(\underline{U}, \underline{\underline{\sigma}}) = -\frac{1}{2}\int_\Omega \text{Tr}\left(\underline{\underline{\sigma}}\,\mathbb{K}^{-1}\underline{\underline{\sigma}}\right)\,d\Omega + \int_\Omega \text{Tr}\left(\underline{\underline{\sigma}}\,\underline{\underline{\varepsilon}}(\underline{U})\right)\,d\Omega - \tau(\underline{U}) \tag{12}$$

for all Kinematically Admissible displacement field (K.A.). $\tau(\underline{U})$ is the lineal form associated with the prescribed forces. We will denote by $\mathcal{U}^{KA}_{KL}$ the following space:

$$\mathcal{U}^{KA}_{KL} = \{(\underline{V}(\text{m}),\text{w}(\text{m}))\ \text{K.A.}/\underline{V}(m) \in H^1(\Sigma_m)\ \&\ w(m) \in H^2(\Sigma_m)\} \tag{13}$$

The space of displacement field of Kirchhoff-Love type Kinematically Admissible to zero (KA0) will be denoted, in the same manner, by $\mathcal{U}^{KA0}_{KL}$.
For $\underline{U}^{KL} \in \mathcal{U}^{KA}_{KL}$ and for all $\underline{\underline{\sigma}}$ such that $\sigma_{zz} = 0$, taking into account the expression of the Hooke tensor under plane stress assumptions, the HR functional reduces to:

$$\begin{aligned} HR(\underline{U}^{KL}, \underline{\underline{\widetilde{\sigma}}}, \underline{\sigma}_c) = &-\frac{1}{2}\int_\Omega\Big(\ \text{Tr}\left(\underline{\underline{\widetilde{\sigma}}}\,\mathbb{K}^{-1}_{cp}\,\underline{\underline{\widetilde{\sigma}}}\right) \\ &+ \left(\underline{\sigma}_c\mathbb{B}^{-1}\,\underline{\sigma}_c\right) + \text{Tr}(\underline{\underline{\widetilde{\sigma}}}^T\,\underline{\underline{\widetilde{\varepsilon}}}(\underline{U}^{KL}))\ \Big)\,d\Omega - \tau(\underline{U}^{KL}) \end{aligned} \tag{14}$$

which, by taking the variation with respect to all $(\underline{\underline{\widetilde{\sigma}}}, \underline{\sigma}_c)$, leads to the following constitutive relations:

$$\underline{\underline{\widetilde{\varepsilon}}} = \mathbb{K}_{cp}^{-1}\,\underline{\underline{\widetilde{\sigma}}}\,,\ \underline{\sigma}_c = \underline{0} \tag{15}$$

Taking these expression into accounts, the HR functional reduces to the potential energy of the Kirchhoff-Love theory, i.e:

$$E_p(\underline{U}^{KL}) = \frac{1}{2}\int_{\Omega} \mathrm{Tr}\left(\underline{\underline{\widetilde{\varepsilon}}}\,\mathbb{K}_{cp}\,\underline{\underline{\widetilde{\varepsilon}}}\right)\,d\Omega - \tau(\underline{U}^{KL}) \tag{16}$$

From the expression of the displacement of the Kirchhoff-Love theory, one obtains:

$$\underline{\underline{\widetilde{\varepsilon}}}(\underline{U}^{KL}) = \underline{\underline{\gamma}} + z\underline{\underline{\chi}} \tag{17}$$

where $\underline{\underline{\gamma}}$ and $\underline{\underline{\chi}}$ are defined as follows, and correspond respectively to the tension strain and to the curvature operator of the plate in its deformed state:

$$\left\{\begin{array}{l} \underline{\underline{\gamma}} = \underline{\underline{\varepsilon}}(\underline{V}) \\ \underline{\underline{\chi}} = \underline{\underline{\mathrm{grad}}}(\underline{\mathrm{grad}}_m\, w) \end{array}\right. \tag{18}$$

Introducing the previous expression, the potential energy reads:

$$E_p(\underline{V}, w) = \frac{1}{2}\int_{\Sigma} \mathrm{Tr}\left(\langle\mathbb{K}_{cp}\rangle\underline{\underline{\gamma}}\ \underline{\underline{\gamma}} + 2\langle z\mathbb{K}_{cp}\rangle\underline{\underline{\gamma}}\ \underline{\underline{\chi}} + \langle z^2\mathbb{K}_{cp}\rangle\underline{\underline{\chi}}\ \underline{\underline{\chi}}\right)\,dS - \tau(\underline{U}^{KL}) \tag{19}$$

where $\langle a\rangle$ defines the integral of a over the thickness $2h$ of the specimen:

$$\langle a\rangle = \int_{-h}^{h} a\,dz \tag{20}$$

2.3 The equations of the CLT in general forces

The minimization of the potential energy with respect to all $(\underline{V}(m), w(m)) \in \mathcal{U}_{KL}^{KA}$ leads to:

$$\begin{aligned} &\forall(\underline{V}^*(m), w^*(m)) \in \mathcal{U}_{KL}^{KA0}: \\ &\int_{\Sigma} \mathrm{Tr}(\langle\mathbb{K}_{cp}\rangle\underline{\underline{\gamma}}(\underline{V})\ \underline{\underline{\gamma}}(\underline{V}^*) + \langle z\mathbb{K}_{cp}\rangle\underline{\underline{\gamma}}(\underline{V})\ \underline{\underline{\chi}}(w^*) + \\ &\langle z\mathbb{K}_{cp}\rangle\underline{\underline{\chi}}(w)\ \underline{\underline{\gamma}}(\underline{V}^*) + \langle z^2\mathbb{K}_{cp}\rangle\underline{\underline{\chi}}(w)\ \underline{\underline{\chi}}(w^*))\,dS \\ &-\tau(\underline{V}^*(m), w^*(m)) = 0 \end{aligned} \tag{21}$$

Introducing, in the previous equation $\underline{\underline{N}}$ the resultant and $\underline{\underline{M}}$ the moment of the in plane stress as follows:

$$\left\{ \begin{array}{l} \underline{\underline{N}} = \int_{-h}^{h} \underline{\underline{\widetilde{\sigma}}}\, dz =< \mathbb{K}_{cp} > \underline{\underline{\gamma}} + < z\, \mathbb{K}_{cp} > \underline{\underline{\chi}} \\ \underline{\underline{M}} = \int_{-h}^{h} z\, \underline{\underline{\widetilde{\sigma}}}\, dz =< z\, \mathbb{K}_{cp} > \underline{\underline{\gamma}} + < z^2\, \mathbb{K}_{cp} > \underline{\underline{\chi}} \end{array} \right. \tag{22}$$

one obtains the expression of the Principal of Virtual Work for the KL theory:

$$\begin{array}{l} \forall(\underline{V}^*(m), w^*(m)) \in \mathcal{U}_{KL}^{KA0} : \\ \int_\Sigma \mathrm{Tr}\left(\underline{\underline{N}}\,\underline{\underline{\gamma}}(\underline{V}^*) + \underline{\underline{M}}\,\underline{\underline{\chi}}(w^*)\right) dS - \tau(\underline{V}^*(m), w^*(m)) = 0 \end{array} \tag{23}$$

Thus it appears that, if $< z\, \mathbb{K}_{cp} > \neq \underline{0}$, there is a coupling between tension and bending. In order to avoid this coupling engineers make use of stacking layers which are symmetrical with respect to the mid-plane. In what follows, for sake of simplicity, we will consider symmetrical stacking layers only. In this case, the tension and bending problems are decoupled.

Let us consider the case where the plate is loaded on its upper and lower surfaces Σ^+ and Σ^-, is submitted on a part Σ_{l2} of Σ_l by a line density of forces, is clamped on the complementary part Σ_{l1} of Σ_l, with the possibility of normal concentrated load on the angular points P_i of Σ_l, as follows:

- Upper surface Σ^+ : $\sigma^+ = \underline{\tau}_c^+ + p^+ \underline{z}$
- Lower surface Σ^- : $\sigma^- = \underline{\tau}_c^- + p^- \underline{z}$
- Lateral loaded surface Σ_{l2} : $\underline{F} = P\underline{z} + \underline{H}$
- Angular points P_i : $\underline{F} = F_i \underline{z}$ (24)
- Complementary part Σ_{l1} : $\underline{V}(m) = \underline{0}$, $w(m) = 0,$ $w_{,n} = \underline{n}^T.\underline{\mathrm{grad}}_m w = 0$

From the expression of the Principal of Virtual Work, integrated once for $\underline{V}^*(m)$ and twice for $w^*(m)$ leads to the following local equilibrium conditions:

- On mean surface Σ_m:

$$\left\{ \begin{array}{l} \mathrm{div}\left(\underline{\mathrm{div}}_m \underline{\underline{M}}\right) + h\ \mathrm{div}\left(\underline{\tau}_c^+ + \underline{\tau}_c^-\right) + (p^+ + p^-) = 0 \\ \underline{\mathrm{div}}_m\left(\underline{\underline{N}}\right) + (\underline{\tau}_c^+ + \underline{\tau}_c^-) = \underline{0} \end{array} \right. \tag{25}$$

- On loaded surfaceΣ_{l2}:

$$\begin{cases} \underline{\underline{N}}.\underline{n} = \underline{H}, \\ \underline{n}^T.\underline{\underline{M}}.\underline{n} = 0, \\ \underline{\mathrm{div}}_m \underline{\underline{M}}.\underline{n} + (\underline{t}^T.\underline{\underline{M}}.\underline{n}_{,s}) - P = 0 \end{cases} \tag{26}$$

- On the angular points P_i:

$$[\![\underline{t}^T.\underline{\underline{M}}.\underline{n}]\!]_{P_i} = F_i \tag{27}$$

2.4 Formulation of the plate problem

From the previous section, it comes that the CLT problem consists in: *Finding* $(\underline{\underline{N}}, \underline{\underline{M}}) \in L_2(\Sigma_m)$ *and* $(w(m), \underline{V}(m) \in \mathcal{U}_{KL}^{KA}$, *such that:*

- $(w(m), \underline{V}(m))$ are Kinematically admissible i.e.: $\in \mathcal{U}_{KL}^{KA}$

- $\left(\underline{\underline{N}}, \underline{\underline{M}}\right) \in L_2(\Sigma_m)$ are Statically admissible i.e.:

$$\begin{aligned} \forall \quad & (\underline{V}^*(m), w^*(m)) \in \mathcal{U}_{KL}^{KA0} : \\ & \textstyle\int_\Omega \mathrm{Tr}(\langle \mathbb{K}_{cp}\rangle \underline{\underline{\gamma}}(\underline{V})\, \underline{\underline{\gamma}}(\underline{V}^*) + \langle z\mathbb{K}_{cp}\rangle \underline{\underline{\gamma}}(\underline{V})\, \underline{\underline{\chi}}(w^*) \\ & + \langle z\mathbb{K}_{cp}\rangle \underline{\underline{\chi}}(w)\ \underline{\underline{\gamma}}(\underline{V}^*) + \langle z^2\mathbb{K}_{cp}\rangle \underline{\underline{\chi}}(w)\ \underline{\underline{\chi}}(w^*))\, d\Omega \\ & - \tau(\underline{V}^*(m), w^*(m)) =\ 0 \end{aligned} \tag{28}$$

- $(w(m), \underline{V}(m))$ and $\left(\underline{\underline{N}}, \underline{\underline{M}}\right)$ satisfy the constitutive plate relation i.e.:

$$\begin{aligned} \underline{\underline{N}} &= \textstyle\int_{-h}^h \underline{\underline{\sigma}}\, dz =< \mathbb{K}_{cp} > \underline{\underline{\gamma}} + < z\, \mathbb{K}_{cp} > \underline{\underline{\chi}} \\ \underline{\underline{M}} &= \textstyle\int_{-h}^h z\, \underline{\underline{\sigma}}\, dz =< z\, \mathbb{K}_{cp} > \underline{\underline{\gamma}} + < z^2\, \mathbb{K}_{cp} > \underline{\underline{\chi}} \end{aligned} \tag{29}$$

In what follows the previous problem is supposed to be solved and the questions we will look concerns the reconstruction of an approximation of the 3D solution knowing the plate one. As explained in the introduction, this will be done firstly by reconstructing an approximation of the inner 3D solution and secondly by the construction of edge effects. Nevertheless the question of the computation of the K-L solution by means of Finite Element analysis is a difficult one, mainly due to the fact that $w \in H_2(\sigma_m)$, therefore this question has given rise to many works (see e.g [26]).

3 Reconstruction of an approximation of the inner 3D solution from the plate solution

3.1 Reconstruction of a quasi-admissible stress field

For sake of simplicity, we consider the case where $< z\,\mathbb{K}_{cp} >= \underline{0}$, which implies that the tension and bending problems are decoupled. We first aim at constructing a stress field $\underline{\underline{\widehat{\sigma}}}$ satisfies all the equilibrium equations apart the boundary conditions on Σ_l. Let us recall that the 3D inner equations satisfied by the stress are in Ω:

$$\left\{ \begin{array}{l} \underline{\text{div}}\ \underline{\underline{\widetilde{\sigma}}} + \underline{\sigma}_{c,z} = \underline{0} \\ \text{div}\ \underline{\sigma}_c + \sigma_{zz,z} = 0 \end{array} \right. \tag{30}$$

and on the lower and top surfaces:

$$\begin{array}{rcl} \underline{\sigma}_{c|h} = \underline{\tau}_c^+ & ; & \underline{\sigma}_{c|-h} = -\underline{\tau}_c^- \\ \sigma_{zz|h} = p^+ & ; & \sigma_{zz|-h} = -p^- \end{array} \tag{31}$$

Therefore knowing the in plane stress $\underline{\underline{\widetilde{\sigma}}}$ associated with the plate solution, the equilibrium equation appears as 3 differentials equations of the first order in z. Each of those equations have two limit conditions, one on the lower surface and one on the upper surface of the plate. Therefore, it is possible to integrate these equations, taking into account the limit conditions, if and only if the following conditions are satisfied.

$$\begin{array}{l} \int_{-h}^{h} \left(\underline{\text{div}}_m\ \underline{\underline{\widetilde{\sigma}}} + \underline{\sigma}_{c,z}\right) dz = 0 \\ \int_{-h}^{h} \left(\text{div}_m\ \underline{\sigma}_c + \sigma_{zz,z}\right) dz = \underline{0} \end{array} \tag{32}$$

which, taking the limit conditions into accounts, leads to:

$$\int_{-h}^{h} \underline{\text{div}}_m\ \underline{\underline{\widetilde{\sigma}}}\ dz + \underline{\tau}_c^+ + \underline{\tau}_c^- = \underline{0} \tag{33}$$

Introducing $\underline{\underline{N}}$ this equation corresponds to the equation of equilibrium of the C.L.T. in tension, i.e.:

$$\underline{\text{div}}_m\ \underline{\underline{N}} + \underline{\tau}_c^+ + \underline{\tau}_c^- = \underline{0} \tag{34}$$

and therefore is automaticaly satisfied.

The other equilibrium equation becomes:

$$p^+ + p^- + [z\ \text{div}_m\ \underline{\sigma}_c]_{-h}^{h} - \int_{-h}^{h} z\ \text{div}_m\ \underline{\sigma}_{c,z}\ dz = 0 \tag{35}$$

which, by integrating by parts, is equivalent to:

$$p^+ + p^- + h\left[\mathrm{div}_m\ (\underline{\tau}_c^+ + \underline{\tau}_c^-)\right]_{-h}^{h} + \int_{-h}^{h} z\ \mathrm{div}_m\ \underline{\mathrm{div}}_m\ \underline{\underline{\widetilde{\sigma}}}\ dz = 0 \qquad (36)$$

Introducing $\underline{\underline{M}}$ this equation corresponds to the equation of equilibrium of the C.L.T. in bending, i.e.:

$$p^+ + p^- + h\left[\mathrm{div}_m\ (\underline{\tau}_c^+ + \underline{\tau}_c^-)\right]_{-h}^{h} + \mathrm{div}_m\ \underline{\mathrm{div}}_m \underline{\underline{M}} = 0 \qquad (37)$$

Therefore the following stress distribution $\underline{\underline{\widehat{\widetilde{\sigma}}}}$ satisfies all the equilibrium equations apart the boundary conditions on Σ_l:

$$\begin{cases} \underline{\underline{\widehat{\widetilde{\sigma}}}} = \mathbb{K}_{cp}(\gamma + z\ \chi) \\ \underline{\widehat{\sigma}}_c = -\int_{-h}^{z} \underline{\mathrm{div}}_m\ \underline{\underline{\widetilde{\sigma}}}\ dz\ -\ \underline{\tau}_c^- \\ \widehat{\sigma}_{zz} = -\int_{-h}^{z} \mathrm{div}_m\ \underline{\widehat{\sigma}}_c\ dz\ -\ \underline{p}^- \end{cases} \qquad (38)$$

3.2 Reconstruction of an associated approximation of the displacement field

To estimate these orders of magnitude, one assume that the following relation holds:

$$\begin{cases} \dfrac{\partial a}{\partial m} = o(\dfrac{a}{L}) \\ \dfrac{\partial a}{\partial z} = o(\dfrac{a}{h}) \end{cases} \qquad (39)$$

where L should be of the order of magnitude of the lateral dimension of the plate or at least large compare to its thickness otherwise the use of a plate theory would be meaningless. Taking these a priori estimations into account, its comes from the equilibrium equation that:

$$\begin{cases} \|\underline{\widehat{\sigma}}_c\| = o(\dfrac{h}{L})\|\underline{\underline{\widehat{\widetilde{\sigma}}}}\| \\ \\ |\widehat{\sigma}_{zz}| = o(\dfrac{h^2}{L^2})\|\underline{\underline{\widehat{\widetilde{\sigma}}}}\| \end{cases} \qquad (40)$$

It is to be noted that a first order approximation of the admissible stress field is thus:

$$\begin{cases} \underline{\underline{\widetilde{\widetilde{\sigma}}}} \\ \underline{\sigma}_c = \underline{0} \\ |\sigma_{zz}| = 0 \end{cases} \qquad (41)$$

The first term to be corrected, that is the one with the largest error (i.e. 100%), concerns the non verification of the plane stress condition.

This is related to what is called the plate paradox. The expression of the plate displacement, which gives correct values for the displacement, leads to large error when derived with respect to z. Therefore an additional normal displacement should be added to have a better representation of the displacement. Let's note $w_1(m,z)$ this additional normal displacement, which does not modifies the in plane strain, $w_1(m,z)$ should be such that:

$$\begin{cases} \underline{\underline{\widetilde{\sigma}}}^c = \widetilde{\underline{\underline{\widehat{\sigma}}}} \\ \sigma^c_{zz} = 0 = Cw_{1,z} + \mathrm{Tr}\left[\mathbb{A}\left(\underline{\underline{\widetilde{\varepsilon}}}(\underline{V}) - z\underline{\underline{\mathrm{grad}}}\left(\underline{\mathrm{grad}}_m(w(m))\right)\right)\right] \end{cases} \tag{42}$$

where we denote by the upper indice $\cdot^c$ the field issued from the constitutive equation. Thus:

$$\begin{cases} w_{1,z} = & -\dfrac{1}{C}\mathrm{Tr}\left[\mathbb{A}\left(\underline{\underline{\widetilde{\varepsilon}}}(\underline{V}) - z\underline{\underline{\mathrm{grad}}}\left(\underline{\mathrm{grad}}_m(w(m))\right)\right)\right] \\ w_1 = & \int_{-h}^{z}\dfrac{1}{C(z')}\mathrm{Tr}\left[\mathbb{A}(z')\left(\underline{\underline{\widetilde{\varepsilon}}}(\underline{V}) - z'\underline{\underline{\mathrm{grad}}}\left(\underline{\mathrm{grad}}_m w\right)\right)\right]dz' \\ & +c(m) \end{cases} \tag{43}$$

In the paragraph devoted to the edge effects analysis this modified displacement field is denoted by $\underline{\widehat{U}}$. The following relation holds:

$$\begin{cases} \underline{\underline{\widetilde{\sigma}}}^c & = \left(\widetilde{\mathbb{K}}\left(\underline{\underline{.}}\right) - \dfrac{\mathbb{A}}{C}\mathrm{Tr}\left(\mathbb{A}(\underline{\underline{.}})\right)\right)\left(\underline{\underline{\widetilde{\varepsilon}}}(\underline{V}) - z\underline{\underline{\mathrm{grad}}}_m\left(\underline{\mathrm{grad}}_m(w)\right)\right) \\ & = \widetilde{\mathbb{K}}_{cp}\left(\underline{\underline{\gamma}} + z\underline{\underline{\chi}}\right) \\ \underline{\sigma}_c & = \mathbb{A}(z')\left(\dfrac{1}{2}\underline{\mathrm{grad}}_m(w_1(m,z))\right) \\ \sigma^c_{zz} & = 0 \end{cases} \tag{44}$$

Choosing $h(m)$ of the second order in h ensures that the couple $(\underline{\widehat{U}}, \underline{\underline{\widehat{\sigma}}})$ leads to an error whose order of magnitude is $err = \alpha(h/l)o(\|\underline{\underline{\widetilde{\sigma}}}\|)$. It is to be noted the correction of the normal displacement on the mid plane is thus of the second order, explaining why $w(m)$, as obtained by the plate theory is a very precise approximation of the actual normal displacement. Moreover the admissible stress distribution is a first order approximation of the 3D one.

3.3 Example of an isotropic plate loaded only on its lateral surfaces

For an isotropic plate, we will go further, and show that it is possible to construct, explicitly from the plate solution, 3D displacement and stresses which satisfy all the equations but the lateral boundary conditions. For sake of simplicity, we will consider the case where the plate is loaded on its lateral surface only. We will further assume that the plate is loaded in bending only, the tension part leading to no difficulty.

3.4 Expression and properties of the plate solution for an isotropic plate loaded only on its lateral surfaces

Introducing the Young's modulus E and the Poisson coefficient ν, the 3D constituive relation is:

$$\underline{\underline{\sigma}} = \frac{E}{1+\nu}(\underline{\underline{\varepsilon}} + \frac{\nu}{1-2\nu}\,\mathrm{Tr}(\underline{\underline{\varepsilon}})\,\mathbb{I}_3) \tag{45}$$

and thus, $\mathbb{K}_{cp}$ takes the following from:

$$\underline{\underline{\widetilde{\sigma}}} = \mathbb{K}_{cp}(z\;\chi) = \frac{E}{1+\nu}(\underline{\underline{\widetilde{\varepsilon}}} + \frac{\nu}{1-\nu}\,\mathrm{Tr}(\underline{\underline{\widetilde{\varepsilon}}})\,\mathbb{I}_2) \tag{46}$$

$\mathbb{K}_{cp}$ being homogeneous, the following relations holds:

$$\underline{\underline{M}} = \int_{-h}^{h} z\,\underline{\underline{\widetilde{\sigma}}}\,dz =< z^2\,\mathbb{K}_{cp} > \underline{\underline{\chi}} = \frac{2h^3}{3}\mathbb{K}_{cp}\underline{\underline{\chi}} \tag{47}$$

and thus:

$$\underline{\underline{\widetilde{\sigma}}} = \frac{3}{2h^3}\,z\,\underline{\underline{M}} \tag{48}$$

where $\underline{\underline{M}}$ satisfy the following equation:

$$\text{On }\;\Sigma_m\,:\;\mathrm{div}\left(\underline{\mathrm{div}}_m\underline{\underline{M}}\right) = 0 \tag{49}$$

with:

$$\underline{\underline{M}} = -\frac{2h^3}{3}\frac{E}{1+\nu}\left(\underline{\underline{\mathrm{grad}}}_m(\underline{\mathrm{grad}}_m(w(m)) + \frac{\nu}{1-\nu}\triangle_m(w)\mathbb{I}_2\right) \tag{50}$$

and thus that:

$$\underline{\mathrm{div}}_m(\underline{\underline{M}}) = -\frac{2h^3}{3}\frac{E}{(1+\nu)(1-\nu)}\left(\underline{\mathrm{grad}}_m(\triangle_m(w)\right) \tag{51}$$

3.5 Iterative reconstruction of the 3D expression of the displacement and stresses: principle and notations

The initial plate solution is denoted by a left upper index ${}^0\cdot$. More generally the i^{th} iterate is denoted by a left upper index ${}^i\cdot$. Each iterate solution is constituted by three fields:

- A displacement field ${}^{i}\underline{U}$ with: ${}^{i}\underline{\underline{\sigma}} = \underline{\underline{\varepsilon}}({}^{i}\underline{U})$
- A stress field associated through the 3D constitutive relation: ${}^{i}\underline{\underline{\sigma}}^{c}$
- A stress field which satify the 3D equilibrium equations: ${}^{i}\widehat{\underline{\underline{\sigma}}}$

The later is constructed starting from the an in-plane stress field ${}^{i}\widehat{\widetilde{\underline{\underline{\sigma}}}}$ as follows:

$$\left\{ \begin{array}{l} {}^{i}\widehat{\underline{\sigma}}_{c} = - \int_{-h}^{z} \underline{\mathrm{div}}_{m} \; {}^{i}\widehat{\widetilde{\underline{\underline{\sigma}}}} \; dz \\ {}^{i}\widehat{\sigma}_{zz} = - \int_{-h}^{z} \mathrm{div}_{m} \; {}^{i}\widehat{\underline{\sigma}}_{c} \; dz \end{array} \right. \tag{52}$$

In order that this field satisfy also the top and bottom conditions of equilibrium, the following conditions have to be satisfied, this condition will be enforced at each stage, in particular to fix the undetermined fields which results form any integration:

$$\left\{ \begin{array}{l} \int_{-h}^{h} \left(\underline{\mathrm{div}}_{m} \; {}^{i}\widehat{\widetilde{\underline{\underline{\sigma}}}} \right) dz = \underline{0} \\ \int_{-h}^{h} \left(\mathrm{div}_{m} \; {}^{i}\widehat{\underline{\sigma}}_{c} \right) dz = 0 \end{array} \right. \tag{53}$$

Once these constructions are performed (if possible), one estimates the order of magnitude of error on the components of the stress field; i. e.:

$$\left\{ \begin{array}{l} \| {}^{i}\widetilde{\underline{\underline{\sigma}}}^{c} - {}^{i} \widehat{\widetilde{\underline{\underline{\sigma}}}} \| \\ \| {}^{i}\underline{\sigma}_{c}^{c} - {}^{i} \widehat{\underline{\sigma}}_{c} \| \\ | {}^{i}\sigma_{zz}^{c} - {}^{i} \widehat{\sigma}_{zz} | \end{array} \right. \tag{54}$$

The term which corresponds to the largest magnitude of error will be used to construct the next iterate of the displacement field, and so on. To estimate these orders of magnitude, let us recall that we make use of the following assumptions:

$$\left\{ \begin{array}{l} \dfrac{\partial a}{\partial m} = o(\dfrac{a}{L}) \\ \\ \dfrac{\partial a}{\partial z} = o(\dfrac{a}{h}) \end{array} \right. \tag{55}$$

It is expected that this largest error of magnitude ${}^{i}err = \alpha (h/l)^{i} o(\| {}^{0}\widetilde{\underline{\underline{\sigma}}}^{c} \|)$. where α is of the order of unity. The process is stopped when $\alpha = 0$.

First order correction The initial ^{0} displacement is the one deduced from the plate theory, i. e.:

$${}^{0}\underline{U} = {}^{0} w(m) \underline{z} - z \, \underline{\mathrm{grad}}_{m} \left({}^{0}w(m) \right) \tag{56}$$

where ${}^{0}w(m) = {}^{P}w(m)$ being the normal displacement of the mid surface obtained by the plate theory. Therefore, the stress field deduced from the constitutive relation is:

$$\begin{cases} {}^0\underline{\underline{\tilde{\sigma}}}^c = -z\dfrac{E}{1+\nu}\left(\underline{\underline{\text{grad}}}_m\left(\underline{\text{grad}}_m({}^0w(m)\right) + \dfrac{\nu}{1-2\nu}\triangle_m({}^0w)\mathbb{I}_2\right) \\ {}^0\underline{\sigma}_c^c = \underline{0} \\ {}^0\sigma_{zz}^c = -z\dfrac{E\nu}{(1+\nu)(1-2\nu)}\triangle_m({}^0w) \end{cases} \tag{57}$$

The Admissible stress field ${}^0\underline{\underline{\hat{\sigma}}}$ is deduced from the in plate stress field associated with the plate solution (see equation 52), leading to an error ${}^0err = o(\|{}^0\underline{\underline{\tilde{\sigma}}}^c\|)$:

$$\begin{cases} {}^0\underline{\underline{\hat{\hat{\sigma}}}} = \dfrac{3}{2h^3}\,z\,\underline{\underline{M}} = -z\dfrac{E}{1+\nu}\left(\underline{\underline{\text{grad}}}_m(\underline{\text{grad}}_m({}^0w(m)) + \dfrac{\nu}{1-\nu}\triangle_m({}^0w)\,\mathbb{I}_2\right) \\ {}^0\underline{\hat{\sigma}}_c = \dfrac{1}{2}(z^2-h^2)\dfrac{E}{(1+\nu)(1-\nu)}\left(\underline{\text{grad}}_m(\triangle_m({}^0w)\right) \\ {}^0\hat{\sigma}_{zz} = -\int_{-h}^{z}\text{div}_m\ {}^0\underline{\hat{\sigma}}_c\ dz = 0 \end{cases} \tag{58}$$

The first term to be corrected, that is the one with the largest error (i.e. 100%), concerns the non verification of the plane stress condition. This is again related to what is called the plate paradox. The expression of the plate displacement, which gives correct values for the displacement, leads to large error when derived with respect to z. Therefore an additional normal displacement should be added to have a better representation of the displacement. Let's note $w_1(m,z)$ this additional normal displacement, which does not modifies the in plane strain, $w_1(m,z)$ is such that:

$$\begin{aligned} {}^1\sigma_{zz}^c &= 0 = C^1\varepsilon_{zz}^c + \text{Tr}(\mathbb{A}\,{}^0\underline{\underline{\tilde{\varepsilon}}}_c) \\ &= \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)w_{1,z} - z\nu\ \text{Tr}\left[\underline{\underline{\text{grad}}}_m\left(\underline{\text{grad}}_m({}^0w(m)\right)\right]\right) \end{aligned} \tag{59}$$

thus:

$$w_{1,z} = z\frac{\nu}{1-\nu}\triangle_m({}^0w) \tag{60}$$

and:

$$w_1 = \frac{z^2}{2}\frac{\nu}{1-\nu}\triangle_m({}^0w) + c(m) \tag{61}$$

The stress associated with the modified displacement field satisfy:

$$\begin{cases} {}^1\underline{\underline{\widetilde{\sigma}}}^c & = -z\dfrac{E}{1+\nu}\left(\underline{\underline{\text{grad}}}_m\left(\underline{\text{grad}}_m({}^0w) + \dfrac{\nu}{1-\nu}\triangle_m({}^0w)\,\mathbb{I}_2\right)\right. \\ {}^1\underline{\sigma}_c^c & = \dfrac{E}{2(1+\nu)}\underline{\text{grad}}_m(w_1) \\ & = \dfrac{E}{2(1+\nu)}\left(z^2\dfrac{\nu}{2(1-\nu)}\underline{\text{grad}}_m\left(\triangle_m({}^0w)\right) + \underline{\text{grad}}_m c(m)\right) \\ {}^1\sigma_{zz}^c & = 0 \end{cases} \tag{62}$$

One can notice that the first order displacement correction leads to the 3D in-plane stress which is equal to the one associated with the plate solution (see equation 58). This is because this field satisfies ${}^1\sigma^c_{zz} = 0$. The Admissible stress field ${}^1\underline{\underline{\widehat{\sigma}}}$ is thus equal to the one deduced from the plate equation, i.e.:

$$\begin{cases} {}^1\underline{\underline{\widehat{\sigma}}} = -z\dfrac{E}{1+\nu}\left(\underline{\underline{\text{grad}}}_m\left(\underline{\text{grad}}_m({}^0w(m)) + \dfrac{\nu}{1-\nu}\triangle_m({}^0w)\,\mathbb{I}_2\right)\right. \\ {}^1\underline{\widehat{\sigma}}_c = \dfrac{1}{2}(z^2-h^2)\dfrac{E}{(1+\nu)(1-\nu)}\left(\underline{\text{grad}}_m(\triangle_m({}^0w))\right) \\ {}^1\widehat{\sigma}_{zz} = 0 \end{cases} \tag{63}$$

The error in stresses after the first order correction is thus such that ${}^1err = (h/l)o(\|{}^0\underline{\underline{\widetilde{\sigma}}}^c\|)$.

Second order correction In order to decrease the error, one should add a second order modification $\underline{V}_2(m,z)$ to the in plane displacement field. This field, which should be odd in z to preserve the bending nature of the solution, is constructed such that:

$$\begin{aligned} &{}^1\underline{\widehat{\sigma}}_c = \frac{1}{2}(z^2-h^2)\frac{E}{(1+\nu)(1-\nu)}\left(\underline{\text{grad}}_m(\triangle_m({}^0w)\right) \\ &= {}^2\underline{\sigma}_c^c = \frac{E}{2(1+\nu)}\left(\frac{z^2}{2}\frac{\nu}{1-\nu}\underline{\text{grad}}_m(\triangle_m({}^0w)) + \underline{\text{grad}}_m c(m) + \underline{V}_{2,z}\right) \end{aligned} \tag{64}$$

Thus:

$$\begin{cases} \underline{\text{grad}}_m c(m) = -h^2\dfrac{1}{1-\nu}\left(\underline{\text{grad}}_m(\triangle^0_m(w)\right) \\ \underline{V}_{2,z}(m,z) = z^2\dfrac{2-\nu}{2(1-\nu)}\,\underline{\text{grad}}_m(\triangle_m(w)) \end{cases} \tag{65}$$

and:

$$\begin{cases} c(m) = -\dfrac{h^2}{1-\nu}\triangle_m({}^0w) + a \\ \underline{V}_2(m,z) = z^3\dfrac{2-\nu}{6(1-\nu)}\ \underline{\text{grad}}_m\left(\triangle_m(w)\right) \end{cases} \quad (66)$$

The constant term a corresponds to a rigid body motion whose possibility is already included in $({}^0w)$, can be taken equal to 0. In what follow we will note $w = {}^0w$ Therefore, after this second iteration the reconstructed displacement field is:

$$\begin{aligned} \underline{U}(m,z) = &\left(w(m) + \frac{1}{1-\nu}(\nu\frac{z^2}{2} - h^2)\ \underline{\text{grad}}_m(\triangle_m(w))\right)\underline{z} \\ &+ \underline{\text{grad}}_m\left(-zw(m) + z^3\frac{2-\nu}{6(1-\nu)}\triangle_m(w)\right) \end{aligned} \quad (67)$$

Taking into account the relation $\triangle_m(\triangle_m w) = 0$, this displacement field leads to the following stress distribution:

$$\begin{cases} \underline{\underline{\tilde{\sigma}}}^c = & \dfrac{E}{1+\nu}\Big[\underline{\underline{\text{grad}}}_m\left(\underline{\text{grad}}_m(-zw(m) + z^3\dfrac{2-\nu}{6(1-\nu)}\triangle_m(w))\right) \\ & +z\dfrac{\nu}{1-\nu}\triangle_m(w)\,\mathbb{I}_2\Big] \\ \underline{\sigma}_c^c = & \dfrac{1}{2}(z^2-h^2)\dfrac{E}{(1+\nu)(1-\nu)}(\underline{\text{grad}}_m(\triangle_m(w)) \\ \sigma_{zz}^c = & 0 \end{cases} \quad (68)$$

This field is also statically admissible since $\underline{V_2}$ is divergence free. Therefore this displacement field satisfies all the 3D equations but the lateral prescribed conditions. Unfortunately that does not mean that the reconstructed solution is exact. This is because as shown in [24] the value of $w(m)$ determined by a plate solution is only a second order approximation of the exact one.

4 Edge effects analysis

In this part, it is shown that, with an additional analysis of the edge area, it is possible, starting from the C.L.T. solution, to evaluate the main part of the 3D edge effect. This leads, on the whole structure, to an estimation of the 3D displacement and stress fields with a relative local error of the order